Soliton molecules in trapped vector nonlinear Schrödinger systems

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We propose a method to build a great variety of stable multisoliton ''molecules'' with coupled light beams in Kerr graded index (GRIN) media or atomic mixtures of Bose-Einstein condensates. We present a general theory and discuss several specific cases, including two-, three-, and four-atom molecules made up of Gaussian modes or vortices. A three-dimensional example is also presented.

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Since the begining of its history, physics has studied simple objects and the ways in which they arrange to form more complex structures. Some remarkable successes include the atomic theory of matter, the structure of nucleus in terms of protons and neutrons and the substructure of nucleons in terms of quarks, to cite a few examples.

Elementary robust objects made of light have been known since 1970s. In fact, *spatial optical solitons*—self-trapped states of light with particlelike properties—have attracted a considerable attention during recent years as possible building blocks of all-optical switching devices where light is used to guide and manipulate light itself [1,2]. Robust solitonic structures appear also in Bose-Einstein condensation, where the dilute quantum gas supports robust structures such as one-dimensional dark solitons [3] or bright solitons [4].

In nonlinear optics, the robust nature of spatial optical solitons [2] allows one to draw an analogy with atomic physics, treating spatial solitons as ''atoms of light.'' Furthermore, when several light beams are combined to produce *a vector soliton*, this process can be viewed as the formation of composite states or ''molecules of light.''

The achievement of stable multisoliton states, with an arbitrary number of ''atoms,'' has been one of the goals of contemporary research in the field of nonlinear optics. Many structures of this type have been studied, for example, spiraling solitons [5], dipole and multipole vector solitons [6–9], self-trapped ring beams [10], rotating propeller solitons [11], and rotating optical soliton clusters [12]. Although these studies have provided a better understanding of different aspects of soliton dynamics, it is clear that *most of these structures are unstable* (i.e., they break due to different instabilities after a sufficiently long time). The most remarkable exception is that of dipole mode vector solitons [6,7] in saturable media. However, the multipole vector solitons, which have a larger number of atoms, are also unstable [8].

In multicomponent Bose-Einstein condensates (BEC), simple nonlinear stationary solutions have been described [13–15], but no attempt has been made to use them to build more complex molecules.

In this paper, we describe and analyze in detail a method to build ''soliton molecules'' with several different properties: (i) an arbitrary number of atoms can be used to build these molecules; (ii) the molecules exist in two-dimensional (2D) and three-dimensional scenarios thus providing the first soliton molecules, shown to exist for $D=3$ (all previous analyses correspond to 2D systems [6–12]); (iii) they are *stable* for any number of constituents; (iv) the molecules can be built from different types of atoms (i.e., nodeless, vortices, . . .); and (v) exist in systems with the simplest type of nonlinearity, the cubic one. Thus, our results can be applied to Kerr media in nonlinear optics and to Bose-Einstein condensed gases.

The model. We will consider a system of N complex fields $u_1(t,\mathbf{r}), u_2(t,\mathbf{r}), \ldots, u_N(t,\mathbf{r})$, ruled by the equations

$$i\partial_t u_j(t,\mathbf{r}) = \left[-\frac{1}{2}\Delta + V(\mathbf{r}) + U_j(t,\mathbf{r}) \right] u_j(t,\mathbf{r}), \qquad (1)$$

for $j=1,\ldots,N$. The coupling term is given by $U_j(t,\mathbf{r}) = \Sigma_k g_{jk}|u_k(t,\mathbf{r})|^2$ with $g_{jk}\in\mathbb{R}$. Equations (1) are a set of nonlinear Schrödinger equations (NLSE) which in BEC problems describe multicomponent systems, u_j being the wave functions for each of the atomic species involved [13–15]. In optics these equations describe the incoherent interaction between the slowly varying envelopes of the electric field in paraxial beams in Kerr media. We choose $V(\mathbf{r}) = \mathbf{r}^2/2$, which corresponds to an isotropic magnetic trapping in BEC and to a GRIN fiber in the optical case.

Single component case: Soliton atoms. Let us first consider the scalar case ($N=1$). Solitons or stationary solutions of Eq. (1) in the scalar case have the form $u(t,\mathbf{r}) = \phi_\mu(\mathbf{r})e^{i\mu t}$ and satisfy

$$\mu\phi_\mu = -\frac{1}{2}\Delta\phi_\mu + \frac{1}{2}\mathbf{r}^2\phi_\mu + g|\phi_\mu|^2\phi_\mu . \qquad (2)$$

For a fixed norm $\|\phi\|_{L^2} := (\int|\phi|^2 dV)^{1/2}$, any solution to Eq. (2) will be a valid soliton atom for our purposes. The simplest case corresponds to a nodeless ground state solution. In the strong interaction case, its shape is close to the Thomas-Fermi solution, and in the small interaction case (as it happens in nonlinear optics) it is close to a Gaussian function.

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Many other complex stationary solutions to Eq. (2) are possible, such as vortices. These objects will play the role of atoms in what follows.

A relevant property is that any function of the form

$$u(\mathbf{r},t)=\phi_\mu(\mathbf{r}-\mathbf{R}(t))e^{i[\mu t+\theta(\mathbf{r},t)]} \tag{3}$$

is a solution of the scalar time-dependent NLSE, provided $\mathbf{R}(t)$ satisfies $(d^2/dt^2)\mathbf{R}+\mathbf{R}=0$ and $\theta(\mathbf{r},t)=(\mathbf{r},(d/dt)\mathbf{R})+f(t)$, where $f(t)=\int_0^t([(d/dt)\mathbf{R}]^2-\mathbf{R}^2)dt$ [16]. This means that exact scalar time-dependent solutions, whose centers evolve according to harmonic-oscillator-type equations and preserve the shape of the stationary solution during evolution, can be built. However, if $N\geqslant 2$, the property described previously does not hold, in general, because of the cross interaction. In a general situation, the pulses collide and lose their individuality. Here we want to give some ideas on how to overcome this problem and build stationary nontrivial vector solitons of Eqs. (1).

Formalism for the multicomponent case. Let us define the modulus n_j and phase ϕ_j of each wave packet through $u_j=\sqrt{n_j}\exp(i\varphi_j)$. Let us define also the center of mass of species j, $\mathbf{R}_j(t)=\int dV\, n_j\mathbf{r}$, and their total momenta $\mathbf{P}_j(t)=\int dV\, n_j\boldsymbol{\nabla}\varphi_j$, whose evolution laws are

$$\frac{d}{dt}\mathbf{R}_j=\mathbf{P}_j, \tag{4}$$

$$\frac{d}{dt}\mathbf{P}_j=-\mathbf{R}_j+\frac{1}{2}\sum_k g_{jk}\mathbf{F}_{jk}. \tag{5}$$

The first term in the right hand side of Eq. (5) corresponds to the external potential, while the nonlinear force is given by

$$\mathbf{F}_{jk}=\int dV(n_k\boldsymbol{\nabla}n_j-n_j\boldsymbol{\nabla}n_k). \tag{6}$$

If n_j for $j=1,\ldots,N$ have small enough overlapping, it is possible to argue that $\mathbf{F}_{jk}$ will be a central force. Let us first notice that for the scalar case and far from the center of the wave packet, the self-interaction is small and u can be described by the linear theory to be $n(\mathbf{r})\propto e^{-r^2}[r^{2m}+O(r^{2m-2})]$ for $r\to\infty$. In the multicomponent case, if the wave packets are separated, then the overlappings between the u_j's will be small and $\boldsymbol{\nabla}n_j=-2(\mathbf{r}-\mathbf{R}_j)+O(|\mathbf{r}-\mathbf{R}_j|^{-1})$. Then, we get

$$\mathbf{F}_{jk}=2(\mathbf{R}_j-\mathbf{R}_k)\int dV\, n_jn_k+O\left(\frac{1}{|\mathbf{R}_k-\mathbf{R}_k|}\right), \tag{7}$$

i.e., if the wave packets are separated, in the leading order the intermode force is central.

Let us evaluate the factor $\int dV\, n_jn_k$, for illustrative purposes, for the case when $u_{j,k}$ are solutions of linear problem $(g_{jk}=0)$ of the particular type, $n_j=N_j|\mathbf{r}-\mathbf{R}_j|^{2m_j}e^{-(\mathbf{r}-\mathbf{R}_j)^2}$, where N_j are constants

$$\int dV\, n_jn_k=\frac{\pi}{2}N_jN_ke^{-(1/2)Q_{jk}^2}K_{m_j,m_k}(Q_{jk}), \tag{8}$$

where $\mathbf{Q}_{jk}=\mathbf{R}_j-\mathbf{R}_k$, $Q_{jk}=|\mathbf{Q}_{jk}|$, and K_{m_j,m_k} are polynomial factors, the lowest order ones for $D=2$ being $K_{0,0}(Q)=1$, $K_{0,1}(Q)=Q^2/4$, and $K_{1,1}(Q)=[2-Q^2+\frac{1}{4}Q^4]/4$. In any case, the specific form of the interaction is less crucial than the fact that the forces $\mathbf{F}_{jk}$ are central.

Two-component case. For the two-component symmetric $(g_{12}=g_{21}=\overline{g})$ case, Eqs. (5) read

$$\frac{d}{dt}\mathbf{P}_1=-\mathbf{R}_1+\frac{1}{2}\overline{g}\mathbf{F}_{12}, \tag{9a}$$

$$\frac{d}{dt}\mathbf{P}_2=-\mathbf{R}_2-\frac{1}{2}\overline{g}\mathbf{F}_{12}. \tag{9b}$$

The most interesting quantity is $\mathbf{Q}\equiv\mathbf{Q}_{12}=\mathbf{R}_1-\mathbf{R}_2$, which gives the separation between the centers of mass of the two components and evolves according to $(\mathbf{F}\equiv\mathbf{F}_{12})$,

$$\frac{d^2\mathbf{Q}}{dt^2}+\mathbf{Q}=\overline{g}\mathbf{F}. \tag{10}$$

As discussed above, if $\mathbf{Q}$ is sufficiently large, the force $\mathbf{F}\propto\mathbf{Q}$ and it can be presented in the potential form $\mathbf{F}=-(\partial/\partial\mathbf{Q})W_{\mathrm{f}}(Q)=-W_{\mathrm{f}}'(Q)\mathbf{Q}/Q$. Then, Eq. (10) reads

$$\frac{d^2\mathbf{Q}}{dt^2}+W_{\mathrm{tot}}'(Q)\frac{\mathbf{Q}}{Q}=0, \tag{11}$$

where $W_{\mathrm{tot}}=\frac{1}{2}Q^2+\overline{g}W_{\mathrm{f}}(Q)$. In the approximation of independent wave packets, $W_{\mathrm{f}}(Q)=\int dV\, n_1n_2$.

Equation (11) has two constants of motion: angular momentum $\mathbf{L}=\mathbf{Q}\times\dot{\mathbf{Q}}$ and energy $E=\frac{1}{2}(\dot{\mathbf{Q}},\dot{\mathbf{Q}})+W_{\mathrm{tot}}(Q)$

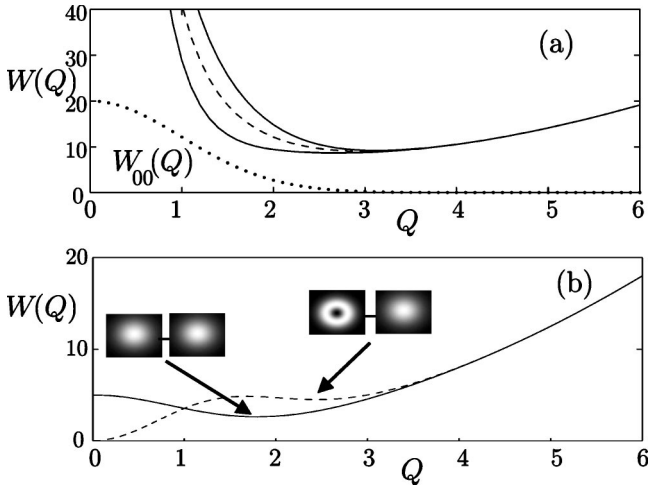

FIG. 1. (a) Potential $W(Q)$ for $g_{jk}=0$ (dashed line) and nonlinear potential $W_f(Q)$ (dotted line) for the interaction of two Gaussian modes. Shown are the total potential for $\overline{g}/2\pi=20$ (upper solid line) and $\overline{g}/2\pi=-20$ (lower solid line). (b) A situation with $L=0$. Solid line shows the total potential for two Gaussian modes with $\overline{g}/2\pi=5$. Dashed line shows the potential for Gaussian plus vortex modes with $\overline{g}/2\pi=20$. The arrows mark the minima corresponding to nonrotating vector molecules.

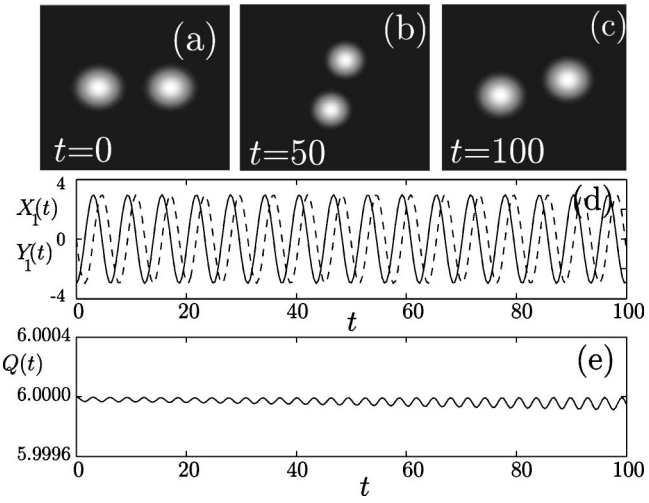

FIG. 2. Evolution of initial data $u_1(\mathbf{r},t)=\phi_0(\mathbf{r}-\mathbf{R}_1(0))e^{3iy}$, $u_2(\mathbf{r},t)=\phi_0(\mathbf{r}+\mathbf{R}_1(0))e^{-3iy}$ with $g_{11}=g_{12}=g_{22}=10$, $\int n_j dV=1$, $\mathbf{R}_1(0)=(3,0)$, and ϕ_0 is the scalar ground state. (a)–(c) Density plots of $n_1(\mathbf{r},t)+n_2(\mathbf{r},t)$ on the spatial region $[-8,8]\times[-8,8]$. (d) Evolution of $\mathbf{R}_1(t)$: $X_1(t)$ (solid) and $Y_1(t)$ (dashed). (e) Evolution of $Q(t)$.

$=\frac{1}{2}\dot{Q}^2+W_{\text{eff}}(Q)$, where $W_{\text{eff}}(Q)=Q^2/2+L^2/(2Q^2)+\bar{g}W_{\text{f}}(Q)$. There will exist an equilibrium distance Q_0 for which the effective potential W_{eff} is minimized. The reason is that the centrifugal contribution is divergent for $Q\simeq 0$, while the trap potential is unbounded for $Q\to\infty$ and the effective nonlinear interaction should decay for large Q and have a maximum finite amplitude (Fig. 1).

In the case of small interaction, $Q_0^{\text{lin}}\simeq L^{1/2}$ leads to a rotating solution of Eqs. (1), provided the distances between the components are kept large enough. For larger nonlinear terms, Q_0 will deviate from Q_0^{lin} ($Q_{eq}>Q_0$ for $g_{jk}>0$ and $Q_{eq}<Q_0$ for $g_{jk}<0$) [Figs. 1(a),1(b)]. When $L=0$, the combination of the trap force and the nonlinear term may also have minima [Fig. 1(c)] corresponding to nonrotating soliton molecules.

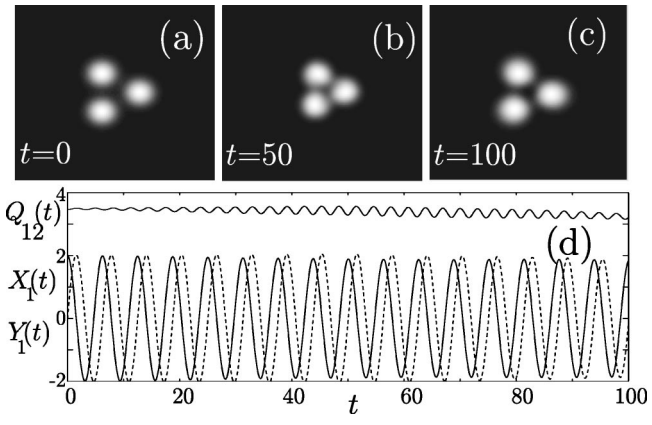

FIG. 3. Evolution of a three-component system with $u_j(\mathbf{r},0)=(1/\sqrt{\pi})e^{-[(x-x_j)^2+(y-y_j)^2]/2}e^{i(v_{xj}x+v_{yj}y)}$, $(x_j,y_j)=d_0(\cos(2\pi j/3),\sin(2\pi j/3))$, $(v_{xj},v_{yj})=d_0(\sin(2\pi j/3),-\cos(2\pi j/3))$ for $j=0,1,2$. Parameter values: $g_{ij}=3, d_0=2, \int n_j dV=1$. (a)–(c) Density plots of $\Sigma_{j=1}^3 n_j(\mathbf{r},t)$. (d) Evolution of $\mathbf{R}_1$: $(X_1(t),Y_1(t))$ and $Q_{12}(t)$.

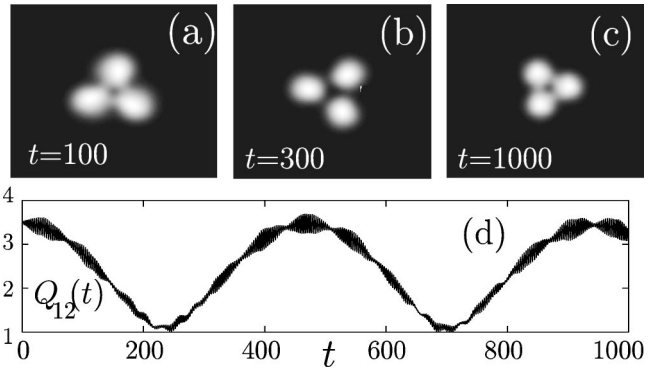

FIG. 4. Same as Fig. 3 but with $g_{ij}=5$. (a–c) Density plots of $\Sigma_{j=1}^3 n_j(\mathbf{r},t)$. (d) Long-time evolution of $Q_{12}(t)$.

From W_{eff} we may also obtain the frequency of small oscillations around Q_0, which is

$$\Omega^2=1+3L^2/Q_0^4+\bar{g}W_f''(Q_0). \tag{12}$$

This is a precise prediction of the theory which will be used to verify its quantitative validity.

Examples of soliton molecules. Let us now present several examples of the soliton molecules discussed previously. First we have studied the case of a pair of weakly interacting soliton atoms. The results, obtained with a symplectic second order in time split-step integrator, are summarized in Fig. 2 where it is seen how the small interaction induces only small oscillations of $Q(t)$ (without distortion of the wave packets), whose dominant fast frequency agrees with the prediction of Eq. (12).

In Fig. 3, three Gaussian solitons interact more strongly due to the larger number of components and the smaller distance between the beams. In this case, the oscillations of the distances between components Q_{ij} remain small [Fig. 3(d)] although some oscillations of the positions of the beams are appreciable [Figs. 3(a)–3(c)]. This leads to a multiscale behavior with a fast frequency given by Eq. (12). Other slower frequencies related to the details of the interaction appear, whose detailed analysis will be the subject of future work. Although this particular configuration is stable, the present

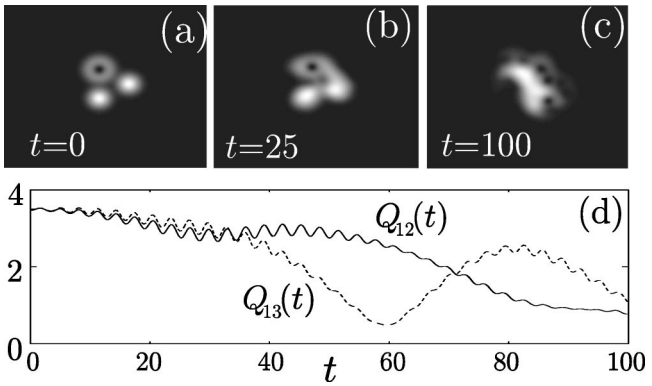

FIG. 5. Same as Fig. 4 but with a Gaussian soliton replaced by a vortex soliton. (a)–(c) Density plots of $\Sigma_{j=1}^3 n_j(\mathbf{r},t)$. (d) Evolution of $Q_{12}(t)$ (solid) and $Q_{13}(t)$ (dashed).

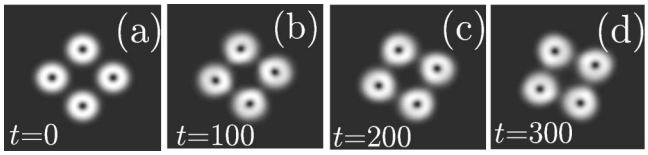

FIG. 6. Evolution of four-vortex solitons placed initially at $(4,0),(0,4),(-4,0),(0,-4)$ and orthogonal speeds of equal moduli. Interaction is $g=10$. (a)–(d) Density plots of $\Sigma_{j=1}^{4} n_j(\mathbf{r},t)$. The system remains stable after the fifty full rounds that take place from $t=0$ to $t=300$.

example is a three-body problem for which many behaviors are possible, such as stable solutions, resonances, chaos, etc. In fact, if the values of the nonlinear coefficient are increased, the beams deform more strongly and the intermode distances $Q(t)$ suffer strong oscillations [Fig. 4(d)] although the structure remains stable after long periods of time, containing about one hundred revolutions of the soliton around the center.

Replacing one of the Gaussian solitons by a vortex soliton (Fig. 5), the asymmetry and the longer interaction range of the vortex soliton (for which $n \sim r^2 e^{-r^2}$) make this configuration unstable and the initial configuration is destroyed after a few rounds.

If only well-separated vortex solitons are used as atoms, stable evolutions can be obtained (Fig. 6). The stability of the four-vortex molecule, shown in Fig. 6, confirms that our scheme is useful to build soliton molecules with different types of atoms.

We have also analyzed several soliton molecules in three spatial dimensions. In Fig. 7, we summarize the results for a stable configuration made up of two weakly interacting Gaussian solitons. The evolution for long times shows that the intercomponent distance $Q(t)$ suffers only small oscillations [Fig. 7(f)] which manifest on the plots where more

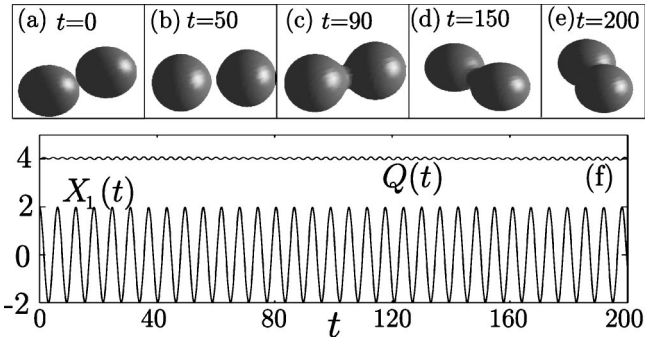

FIG. 7. Weak interaction of two Gaussian solitons $u_1(\mathbf{r},0)=(1/\pi^{3/4})e^{-[(x-2)^2+y^2+z^2]/2}e^{-2iy}$, $u_2(\mathbf{r},0)=(1/\pi^{3/4}) \times e^{-[(x+2)^2+y^2+z^2]/2}e^{2iy}$ with $g_{jk}=10$. (a)–(e) Isosurface plots of $n_1(\mathbf{r},t)+n_2(\mathbf{r},t)$ for $n_1+n_2=0.01$. (f) Evolution of the intermode distance $Q(t)$ and $X_1(t)$.

interaction is apparent [Figs. 7(c),7(d)]. The stability of this three-dimensional object is remarkable since it corresponds to the first soliton molecule described in three dimensions. More complex molecules with larger number of components can be built in three dimensions.

Conclusions. In this paper, we have presented several soliton molecules built up from scalar solitons of the trapped NLSE. They exist for $D=2$ and $D=3$, are stable, can be built with any number of lobes, and can be made from different types of elementary components, such as Gaussian-like modes, vortices, etc.

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